Adjoint matrix: The adjoint of a matrix A is the transpose of the cofactor matrix of A . It is denoted by adj A. An adjoint matrix is also called an adjugate matrix.

Example: Find the cofactor matrix of $\mathbf{A}$ given that $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$.

Solution: First find the cofactor of each element.

$$A_{11} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24 \qquad A_{12} = -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = 5 \qquad A_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4$$

$$A_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -12 \qquad A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \qquad A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$A_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \qquad A_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5 \qquad A_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$$

The cofactor matrix is thus $\begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$.

Now

$Adj\ A = transpose\ of\ cofactor\ matrix\ of\ A$

$\Rightarrow Adj\ A = transpose\ of \begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix} \Rightarrow Adj\ A = \begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$

What is an inverse matrix?

When working with numbers such as 3 or –5, there is a number called the multiplicative inverse that you can multiply each of these by to get the identity 1.

In the case of 3, that inverse 1/3, and in the case of –5, it is –1/5.

The inverse of a matrix A is a matrix that, when multiplied by A results in the identity. The notation for this inverse matrix is A^{-1}.

The product of A and its inverse is the identity:

$$\begin{bmatrix} -3 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{5} \\ 1 & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

matrix A — matrix A^{-1} — 2 x 2 identity matrix

Does every matrix have an inverse?

Thinking about the number 0, there is no number you can multiply it by to get 1. So, the number 0 has no multiplicative inverse. Similarly, not every matrix has an inverse.

A matrix 'A' has an inverse if

(i) A is square matrix
(ii) A is non -*singular matrix*

Show that $\begin{bmatrix} 8 & 2 \\ 4 & 3 \end{bmatrix}$ is non − singular.

Solution

Let $A = \begin{bmatrix} 8 & 2 \\ 4 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 8 & 2 \\ 4 & 3 \end{vmatrix} = 24 - 8 = 16 \neq 0$$

$\therefore$ *A* is a non-singular matrix

So it has an inverse matrix.

For instance, the matrix $A = \begin{bmatrix} 3 & 8 & 1 \\ -4 & 1 & 1 \\ -4 & 1 & 1 \end{bmatrix}$ is a singular matrix, since

$$|A| = 3(1-1) - 8(-4+4) + 1(-4+4) = 0.$$

If $B = \begin{bmatrix} 2 & 6 & 1 \\ -3 & 0 & 5 \\ 5 & 4 & -7 \end{bmatrix}$ then $|B| = 2(0-20) - (-3)(-42-4) + 5(30-0) = -28 \neq 0.$

So it has no inverse.

Methods for finding inverse matrix:

(i) Adjoint matrix method
(ii) Row elementary operation method

Adjoint matrix method: $A^{-1} = \dfrac{\text{Adj of A}}{\text{Deteminant of A}} = \dfrac{\text{Adj of A}}{|A|}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix} \qquad \text{cofactor matrix} = \begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$$

$$A_{11} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24 \quad A_{12} = -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = 5 \quad A_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4$$

$$A_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -12 \quad A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \quad A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$A_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \quad A_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5 \quad A_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$$

Method 1: The Adjoint Method

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix} \qquad \text{Adjoint} = \begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}, \ |A| = 22$$

$$A^{-1} = \frac{\begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}}{22} = \begin{bmatrix} 1.09 & -.5454 & -.0909 \\ .2273 & .1363 & -.2273 \\ -.1818 & .0909 & .1818 \end{bmatrix}$$

(a) $A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{|A|}.adj(A)$

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}; |A| = 0 - 1(1-9) + 2(1-6) = 8 - 10$$

$|A| = -2 \neq 0$

$$Adj\ A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$A_{11} = (-1)^{1+1}[(2)(1) - (3)(1)] = -1$

$A_{12} = 8,\ A_{13} = -5\ ,\ A_{21} = 1\ ,\ A_{22} = -6$

$A_{23} = 3\ , A_{31} = -1\ ,\ A_{32} = 2\ ,\ A_{33} = -1$

$$\therefore\ A^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}.$$

Problem: Find the inverse of the matrix $A = \begin{bmatrix} 6 & 5 \\ 5 & 4 \end{bmatrix}$ by Adjoint matrix method

ans: $A^{-1} = \begin{bmatrix} -4 & 5 \\ 5 & -6 \end{bmatrix}$

Problem: Find the inverse of the matrix $A = \begin{bmatrix} 7 & 1 \\ -3 & 2 \end{bmatrix}$ by Adjoint matrix method

ans: $A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Problem: Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & -1 \\ -5 & 1 & 1 \end{bmatrix}$ by Adjoint matrix method